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Only the main results are summarized and for the details the reader is referred to the appropriate publications. All references here refer to our publications, a list of which is included at the end of this report. Also included at the end of the report are the abstracts of the publications.

The main focus of this project was the evaluation of high resolution spectrum estimation

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methods with particular emphasis on state space based methods. Of particular interest are finite precision effects (numerical issues), and the statistical efficiency of high resolution methods. The results have in a significant manner helped in the understanding of these methods. The results are divided into and discussed under three categories: Spectrum Estimation, Array processing and Floating Point arithmetic.

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FINAL REPORT

TITLE: FINITE WORD LENGTH CONSIDERATIONS IN HIGH RESOLUTION SPECTRAL ESTIMATION

AUTHOR: BHASKAR D. RAO

INSTITUTION: UNIVERSITY OF CALIFORNIA, SAN DIEGO

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FINAL REPORT

This report summarizes the research results obtained in the past three years, and supported by ~~the above~~ grant from the U.S. Army Research Office. Only the main results are summarized and for the details the reader is referred to the appropriate publications. All references here refer to our publications, a list of which is included at the end of this report. Also included at the end of the report are the abstracts of the publications.

The main focus of this project was the evaluation of high resolution spectrum estimation methods with particular emphasis on state space based methods. Of particular interest are finite precision effects (numerical issues), and the statistical efficiency of high resolution methods. We believe our results have in a significant manner helped in the understanding of these methods. The results are divided into and discussed under three categories: Spectrum Estimation, Array processing and Floating Point arithmetic.

Spectrum Estimation:

In this area, the problem of frequency estimation of sinusoids from time series is considered. Of particular interest is the estimation of frequencies from short and noisy data records. Fourier based methods have many deficiencies and are not appropriate in this context. On the other hand, model based methods have proven to have high resolution capabilty and have attracted a great deal of interest. Numerous model based methods have been developed and applied to the problem of frequency estimation. Of particular interest are Singular Value Decomposition (SVD) based methods which show great promise. Such methods are analyzed in our work with particular attention being paid to state space based methods, i.e. methods developed using a state space representation for the models. The main results in this context are as follows:

1. Evaluation of model based methods is fairly involved, and in a great deal of the existing work, computer simulations have been used to support the methods. In this research, an insightful approach to evaluate the methods has been developed. It was recognized that most

model based methods can be divided into two distinct steps. Step I consists of estimating the parameters of the model from the given data, and Step II consists of extracting the required information, frequency estimates in this case, from the estimated parameters. Both steps are important to the overall quality of the estimate. Step I requires reliable estimation of the parameter set, and Step II requires that the parameter set have low parameter sensitivity. An evaluation of both steps are conducted to obtain insight into SVD based methods. A more extensive discussion of the pros and cons of such an evaluation can be found in [7,13,2,4,1].

2. The two step analysis was first applied to the SVD based method of Tufts and Kumaresan. In this method, the linear predictability of sinusoids is exploited to obtain the frequency estimates. The procedure consists of first estimating the linear prediction parameters using a SVD based approach, and then estimating the frequencies from the roots of the linear prediction polynomial. It was shown that for step II, the parameter sensitivity, a large polynomial along with the minimum norm solution are the key. This allows the redundant roots of the polynomial to play an important role in reducing the sensitivity of the desired roots to perturbations in the coefficients [7,2,4]. In Step I, for reliable estimation, it is shown that the continuity of the generalized inverse and the concept of angle between subspaces play an important role. The continuity concept helps explain the need for a low rank approximation in the estimation process, and the quality of the approximation is appraised using the concept of angle between subspaces [7,2,4].

3. Step I, reliable estimation is further explored in [1] and [3,6]. In [1], it is shown that the least squares problem solved in the Kumaresan-Prony approach can be ill-conditioned leading to degradation in the quality of the estimates. In particular, this provided a theoretical explanation to a strange behavior that was observed by others when the method was used for direction finding, namely that for a particular two source scenario the performance degraded as the separation increased as opposed to improving as is usually the case. In [6,3], the concept of angle between subspaces is further explored. This concept provides a procedure to decide when SVD based methods fail, i.e. gives an indication of the theoretical limits of SVD based

methods. Also it was found that existing SVD based model based methods did not exploit the information in the singular vectors maximally. In [6], a non-linear method starting from the singular vectors was developed. The method consisted of fitting a structured subspace to the space spanned by the principal singular vectors. The subspace fitting method turns out to be better than the existing SVD methods but is computationally more intense. However, the ideas developed provide important insight namely that the existing methods do not necessarily make the best use of the principal singular vectors.

4. The state space methods are studied in [2,13,4]. The approach differs from the Tufts and Kumaresan approach in the fact that it is based on a state space parameterization of the model instead of the polynomial parameterization. The state space methods use a SVD based approach to estimate a state transition matrix, and the frequency estimates are obtained from the eigenvalues of the matrix. The issues related to estimation, step I, turn out to be the same as before and so only parameter sensitivity, Step II. is examined in detail. It is shown in the context of sinusoid frequency estimation that from a parameter sensitivity point of view the best state transition matrix to estimate is an unitary matrix. Procedures to estimate such unitary matrices are then explored. It is shown that the Toeplitz Approximation Method (TAM) and the Direct Data Approximation (DDA), two existing methods do estimate such robust matrices in a reliable manner [2,13,4].

5. In [5], attempts to further improve the estimation procedure are made. The fact that the matrix obtained using TAM is ideally unitary is incorporated. This leads to a constrained least squares problem with a fairly simple solution [5].

6. Interesting connections between state space methods and the Matrix Pencil approach are shown in [18]. It is shown that the state space approach and the Matrix Pencil approach are very closely related [18] in that if the state space method can be viewed as estimating a matrix $F=AB$, then the matrix pencil method amounts to estimating $F=BA$. Note that the nonzero eigenvalues of AB and BA are the same leading to the same frequency estimates. This implies that the performance of the two methods are nearly identical. The superiority of State

Space methods over the polynomial approach, e.g. Tufts and Kumaresan method, follow from the results of Hua and Sarkar who show that Matrix Pencil methods are usually superior or comparable in performance statistically to polynomial methods. In addition to the statistical performance, state space methods are often computationally more desirable. One particularly advantageous feature of state space methods is that extracting the frequency estimate from the model parameters is much simpler.

7. In addition to the sinusoid problem, the state space formulation has wider applicability and these are highlighted in [2,20]. The applications of state space ideas to variety of problems, e.g. estimating parameters of damped exponentials, ARMA power spectrum etc. are discussed. A key outcome of this research is evidence to indicate that the state space formulism offers numerous advantages. For instance, it provides a convenient framework for exposing structure inherent in signal processing problems. Also the flexibity it provides in the parameter domain can be exploited to lower parameter sensitivity. Furthermore reliable methods exist to estimate these robust parameter sets. All these attributes are very beneficial to signal processing algorithms.

Subspace Based Array Processing:

Similar to methods in spectrum estimation, methods based on SVD or Eigen decomposition are popular for the problem of direction finding or the direction of arrival (DOA) estimation problem using a sensor array. In this problem the data consists of the array output at different time instants. The output of the array at any time instant is referred to as a snapshot. Using the snapshots, an estimate of the covariance matrix of the data is obtained. Then an estimate of the DOA is obtained by using an eigendecomposition of the covariance matrix. Such methods are often referred to as Subspace based methods. There are numerous subspace based methods and the purpose of this research was to evaluate their pros and cons. Our research analyzed some of the popular methods, namely MUSIC, ESPRIT, TAM and the Minimum-Norm method, under the assumption that the snapshots were independent and that the data

was circularly Gaussian. Also the estimate of the covariance matrix considered in the analysis is that obtained by taking the average of the outer product of the snapshot vectors. The main results of the analysis are as follows:

1. It is shown that for the linear equispaced sensor array case, ESPRIT can be treated via a state space formulation. The matrix obtained using ESPRIT is shown to be related to the matrix obtained using TAM, by a diagonal transformation. TAM and (Least Squares) ESPRIT are shown to have identical statistical properties. Also Total Least Square ESPRIT and Least Squares ESPRIT are shown to have the same asymptotic mean squared error. Closed form expressions are derived for the mean squared error. Simple closed form expressions are derived for the one and two source case leading to interesting insights [8,14].

2. For the linear equispaced array case, Closed form expressions for the asymptotic mean squared error in the estimate of the direction of arrival for the Minimum-Norm method and Root-Music are also obtained [12,17,9,15]. Root-Music is a variant of MUSIC in that the DOA *estimates are obtained from the roots of* a polynomial obtained by performing a spectral factorization of the null spectrum [9,15]. It is shown that the DOA estimates obtained using Root-Music has a smaller mean squared error compared to the Minimum-Norm method. Also the estimates obtained using Root-MUSIC become significantly better as the array length increases. Simple closed form expressions are derived for the one and two source case. Similar conclusion also apply when Root-MUSIC and ESPRIT are compared. Root-MUSIC appears to be the best of the three under the assumed conditions.

3. It was shown that in the case of Root-MUSIC, though the errors in the DOA estimates were small, the error in the roots was large. This implies that for Root-Music the errors in the roots were mainly radial. This observation is fairly important in that it implies trouble when using a spectral approach. The radial errors result in less well defined peaks. If the roots are close together, due to the large radial error there may not even be two peaks in the spectrum. This in turn results in a apparent loss in resolution and a higher threshold SNR for (spectral) MUSIC (compared to the spectral Minimum-Norm method). Also it is shown that

asymptotically there is no difference between using a spectral approach versus using a rooting procedure. However, for the reasons explained above, rooting is desirable for MUSIC wherever possible.

In summary, the above results provide valuable insight into the above methods in the context of a linear equispaced sensor array and when the sources are uncorrelated or partially correlated. To overcome the problems that arise when the sources are fully correlated, spatial smoothing is employed. In this approach the array is divided into smaller (overlapping) subarrays and the overall covariance estimate obtained by averaging over the covariance estimate obtained from each subarray [19]. The main results in the context of spatial smoothing are as follows.

4. The statistics of the eigenvectors of a spatially smoothed forward-backward covariance matrix are presented [19]. This result is of general interest, and is used to evaluate subspace based array processing methods. The procedure used to obtain these statistics consists of using a first order expansion of the signal space eigenvectors. The approach is fairly general and can be easily generalized to deal with other scenarios, i.e. effect of sensor perturbations, deterministic data model etc [19].

5. The statistics are used to study the effect of spatial smoothing on Root-MUSIC and the Minimum-Norm method. Expressions for the mean squared error in the DOA estimates are derived [19]. The methods are compared and it is found that spatial smoothing is more beneficial to the Minimum-Norm method compared to MUSIC. In our earlier work it was shown that when no smoothing was used, MUSIC was superior to the Minimum-Norm method [12,17]. Here it is found that by properly choosing the number of subarrays, the performance of the Minimum-Norm method can be made comparable to MUSIC [19]. In the case of MUSIC, though the DOA estimates usually tend to deteriorate as the number of subarrays increase, the mean square error in the roots improve. Hence, if spectral MUSIC is used, the peaks will be sharper as the number of subarrays is increased giving the false impression of better resolution.

6. The effect of spatial smoothing on ESPRIT is also studied. Similar to the Minimum Norm method, spatial smoothing is beneficial to ESPRIT and significantly improves the estimates.
7. Some implementation (computer architecture) issues are also considered in [10]. It is shown that since State Space based DOA methods, and ESPRIT essentially require only matrix operations, they are very suitable for systolic/wavefront implementation [10,20].

Finite Precision Floating Point Arithmetic:

The effect of finite precision floating point arithmetic in digital signal processing is not well understood. However with the growing availability of floating point chips, understanding the effect of finite precision floating point arithmetic is an increasingly important problem. Our work in this area is very fundamental, and has the potential of providing insight into many applications, e.g. effect of finite precision floating point arithmetic on digital filters [11,16], effect of finite precision floating point arithmetic on adaptive algorithms etc. The main results are as follows:
1. As a starting point for the study, the problem of digital filters is addressed with particular emphasis on robust state space digital filters. In this context, fixed point arithmetic has been extensively studied. The limited success with analyzing the effect of floating point arithmetic on digital filters has been due to the fact that the roundoff errors are correlated with the signals making the analysis tractable. In our work to overcome this difficulty, a very unique interpretation of floating point arithmetic is provided. It is shown that the inner product $a^t x$, an operation basic to linear time invariant systems, when computed using finite precision floating point arithmetic is equivalent to $(a+\Delta a)^t x$, i.e. the exact inner product of a perturbed a and x. This interpretation in the context of digital filters implies that the roundoff noise can be determined by perturbing the filter parameters. It provides a strong link between coefficient sensitivity and roundoff noise and also makes the mathematics much more tractable.
2. An expression for the variance of the output roundoff noise is derived and shown to depend on the filter parameters, in particular the Observabilty and Controllabilty Grammians. Unlike

in the fixed point case, the variance also depends on the input signal statistics. For the case where double prec' . on accumulation is used, it is shown that the optimal filters are similar to those obtain ' when using fixed point arithmetic.

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PUBLICATIONS

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1. Rao, D. V. B., "An Analysis of the Kumaresan-Prony Algorithm," Proceedings of the Twentieth Annual Asilomar Conference on Signals, Systems and Computers (Pacific Grove, California, November, 1986), pp. 467-471.

2. Rao, B. D., "Sensitivity Analysis of State Space Methods Spectrum Estimation," Proceedings of the IEEE 1987 International Conference on Acoustics, Speech and Signal Processing (Dallas, Texas, April 6-9, 1987), pp. 1517-1520.

3. Rao, B. D., "Quality of Low Rank Approximants Obtained Using SVD and Some Applications," Proceedings of the Twenty-first Asilomar Conference on Signals, Systems and Computers (Pacific Grove, California, November, 1987), pp. 784-788.

4. Rao, B. D., "Numerical Considerations in Spectrum Estimation," (A) Indo-US Workshop on Systems and Signal Processing (Bangalore, India, January 8-12, 1988), pp. 83-84. (B) An expanded version to appear in the Book of Proceedings of the Indo-US Workshop, 1988, to be published by Oxford and IBH Publishing.

5. Arun, K. S. and B. D. Rao, "An Improved Toeplitz Approximation Method," Proceedings of the International Conference on Acoustics, Speech and Signal Processing (New York, New York, April, 1988), pp. 2352-2355.

6. Rao, B. D., "Lowering the Threshold SNR of Singular Value Decomposition Based Methods," Proceedings of the International Conference on Acoustics, Speech and Signal Processing (New York, New York, April, 1988), pp. 2472-2475.

7. Rao, B. D., "Perturbation Analysis of a SVD Based Linear Prediction Method for Estimating the Frequencies of Multiple Sinusoids," IEEE Trans. on Acoustics, Speech and Signal Processing, Vol. 36, No. 7, pp. 1026-1035, July, 1988.

8. Rao, B. D., and K. V. S. Hari, "Performance Analysis of Subspace Based Methods," in Proc. 4th ASSP Workshop on Spectrum Estimation and Modeling, pp. 92-97, August, 1988.

9. B. D. Rao and K. V. S. Hari, "Performance Analysis of Root-Music," Vol. 2, pp. 578-582, Twenty-Second Asilomar Conference on Signals, Systems and Computers (Pacific Grove, California, October, 31 - November 2, 1988).

10. B. D. Rao, "State Space Methods for DOA Estimation," High Speed Computing II, SPIE Symposium (Los Angeles, California, January 15-20, 1989).

11. B. D. Rao, "Roundoff Noise in Floating Point State Space Digital Filters," 1989 International Symposium on Circuits and Systems (Portland, Oregon, May 9-11, 1989).

12. B. D. Rao and K. V. S. Hari, "Statistical Performance Analysis of the Minimum-Norm Method," Vol. 4, pp. 2760-2763, Proceedings of the IEEE 1989 Conference on Acoustics, Speech and Signal Processing (Glasgow, Scotland, May 23-26, 1989).

13. B. D. Rao, "Sensitivity Considerations in State Space Methods for the Harmonic Retrieval Problem," To appear in the IEEE Trans. on Acoustics, Speech and Signal Processing, November, 1989.

14. Bhaskar D. Rao and K. V. S. Hari, "Performance Analysis of ESPRIT and TAM in Determining the Direction of Arrival of Plane Waves in Noise," To appear in the IEEE Trans. on Acoustics, Speech and Signal Processing, December, 1989.

15. Bhaskar D. Rao and K. V. S. Hari, "Performance Analysis of Root-Music," To appear in the IEEE Trans. on Acoustics, Speech and Signal Processing, December, 1989.

16. B. D. Rao, "Roundoff Noise in Floating Point State Space Digital Filters," submitted to IEEE Trans. on Acoustics, Speech and Signal Processing.

17. Bhaskar D. Rao and K. V. S. Hari, "Statistical Performance Analysis of Minimum-Norm Method," IEE Proceedings Part F, Communication, Radar and Signal Processing, Vol. 136, No. 3, pp. 125-134, June, 1989.

18. B. D. Rao, "Relationship between Matrix Pencil and State Space Based Harmonic Retrieval Methods," to appear in the IEEE Trans. on Acoustics, Speech and Signal Processing.

19. B. D. Rao and K. V. S. Hari, "Effect of Spatial Smoothing on the performance of MUSIC and the Minimum-Norm Method," Submitted to IEEE Trans. on Acoustics, Speech, and Signal Processing.

20. B. D. Rao, "State Space Model-Based Parameter Estimation Methods and some Applications," SPIE Proceedings, Vol. 1152, August 8-10, 1989.

WORK IN PREPARATION

1. B. D. Rao and K. V. S. Hari, "Music and Spatial Smoothing: A Statistical Performance Analysis," Twenty-Third Annual Asilomar Conference on Signals, Systems and Computers (Pacific Grove, California, October 30-November 1, 1989).

2. B. D. Rao, "Floating Point Arithmetic and Digital Filters," Twenty-Third Annual Asilomar Conference on Signals, Systems and Computers (Pacific Grove, California, October 30-November 1, 1989).

3. B.D. Rao and K. S. Arun, "Model based processing of Signals: A State Space Approach," submitted to Proc. of IEEE.

APPENDIX

This appendix contains copies of the abstracts of the papers supported by this research grant.

AN ANALYSIS OF THE KUMARESAN-PRONY METHOD*

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ABSTRACT

In this paper, we perform a numerical analysis of the Kumaresan-Prony method, a method for estimating the frequencies of sinusoids in white noise. Like most model based methods, in this approach the reliability of the Estimation procedure and Parameter sensitivity are central to determining its performance. We first examine the parameter sensitivity issue and explain how the excess filter length helps combat numerical sensitivity. We examine the estimation procedure and indicate how the concept of angle between subspaces can be used to determine the quality of estimation. An interesting outcome of the analysis is the explanation provided to the limitation of the Kumaresan-Prony algorithm when applied to the direction finding problem.

I. INTRODUCTION

In this paper we perform a numerical analysis of the Kumaresan-Prony (KP) method. The KP method is a special case of the general Tufts and Kumaresan approach, a SVD based linear prediction method for estimating the frequencies of sinusoids in noise [1]. To begin the analysis we note that the model based methods can in general be divided into two distinct steps as in fig.(1) [2]. Step I consists of estimating a parameter set that describes the model and step II consists of determining the relevant information from the parameter estimates. Both steps are important to the overall success of the method. Ill-conditioning at either step can adversely effect the overall performance of the method and needs to be avoided [2,3].

For the sinusoid problem let us denote the parameter set by $\theta = [\theta_1, \theta_2, \cdots, \theta_r]$ and the quantity of interest are the frequencies ω_i's. An error $\Delta\theta$ in the estimate of θ results in a error $\Delta\omega_i$ in the frequency, where

$$\Delta\omega_i = \sum_{j=1}^{r} \frac{\partial\omega_i}{\partial\theta_j} \Delta\theta_j = \Delta\omega^t \Delta\Theta ,$$

where

$$\Delta\omega = \left[\frac{\partial\omega_i}{\partial\theta_1}, \frac{\partial\omega_i}{\partial\theta_2}, \cdots, \frac{\partial\omega_i}{\partial\theta_r}\right]^t .$$

and $\Delta\Theta$ is defined similarly

$$E\,|\Delta\omega_i|^2 = \Delta\omega^t\, E\left[\Delta\Theta\, \Delta\Theta^t\right] \Delta\omega \leq S\, \|R_\theta\|$$

where

$$S = \|\Delta\omega\|^2 = \sum_{j=1}^{r} \left|\frac{\partial\omega_i}{\partial\theta_j}\right|^2 \quad \text{and} \quad R_\theta = E\left[\Delta\Theta\, \Delta\Theta^t\right]$$

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+Unless mentioned otherwise the norm referred to in this paper is the 2-norm

S represents the sensitivity of the parameter set and R_θ the estimation error covariance. In step I, one attempts to minimize $\|R_\theta\|^{+}$. However for the overall success of the method, ill-conditioning in step II has to be avoided, i.e. S has to be small. Reduced sensitivity also reduces the demands on the estimation process. Though an optimality in both steps is desirable, in practice, this may be hard to achieve. However it is clear that an analysis of the two steps separately can be useful in understanding and reaching the best compromise. In this paper we conduct such a two step analysis of the KP method.

II. BACKGROUND

The observations $y(k)$ consists of the sinusoidal signal $z(k)$ corrupted by white noise $n(k)$, i.e.

$$y(k) = z(k) + n(k), \quad k = 1, 2, \ldots, N \tag{1a}$$

The signal

$$z(k) = \sum_{i=1}^{p} c_i e^{j\omega_i k} , \tag{1b}$$

where c_i's are the complex amplitudes and ω_i the frequencies.

A popular procedure to estimate the frequencies is to use an Linear Prediction (LP) approach [4]. In the absence of noise the sinusoidal signal is exactly predictable, i.e.

$$y(k) - \sum_{i=1}^{p} g_i\, y(k-i) = 0. \tag{2}$$

The zeros of the polynomial formed from the coefficients, $H(z) = 1 - \sum_{i=1}^{p} g_i z^{-i}$, gives an estimate of the frequencies. Using the LP approach the problem of frequency estimation is reduced to that of estimating the coefficients g_i. For the estimation of the coefficients a least square approach can be used. In the presence of noise the procedure is less efficient and improvements can be achieved by using a forward backward prediction approach [5,6]. It can be shown that the same coefficients appear in the backward prediction equation, i.e.

$$y(k) - \sum_{i=1}^{p} g_i^*\, y(k+i) = 0.$$

Using the forward backward approach, the problem reduces to solving the following set of linear equations.

$$\begin{bmatrix} y(p) & y(p-1) & \cdots & y(1) \\ y(p-1) & y(p-2) & \cdots & y(2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y(N-1) & y(N-2) & \cdots & y(N-p) \\ y^*(2) & y^*(3) & \cdots & y^*(p+1) \\ y^*(3) & y^*(4) & \cdots & y^*(p+2) \\ \cdot & \cdot & & \cdot \\ y^*(N-p+1) & y^*(N-p+2) & & y^*(N) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_p \end{bmatrix} = \begin{bmatrix} y(p+1) \\ y(p+2) \\ \cdot \\ y(N) \\ y^*(1) \\ y^*(2) \\ \cdot \\ y^*(N-p) \end{bmatrix}$$

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SENSITIVITY ANALYSIS OF STATE SPACE METHODS IN SPECTRUM ESTIMATION*

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ABSTRACT

In this paper, we examine the issue of parameter sensitivity in spectrum estimation. In particular, state space models are considered and robust coordinate systems for spectrum estimation are identified. For the sinusoid problem it is shown that the ideal parameter set involves estimating a unitary matrix and simple procedures to estimate such a matrix are identified. For the damped sinusoid and the general ARMA spectrum estimation problem it is shown that the balanced coordinates are robust and reliable.

I. INTRODUCTION

In recent years there has been a great deal of interest in model based spectrum estimation methods [1]. They provide a tool to extend the data beyond the observation interval and are capable of providing high resolution spectral estimates. In this paper we conduct a numerical study of some spectral estimation methods with particular emphasis on state space modeling. To begin the analysis we note that the model based methods can in general be divided into two distinct steps as in fig (1) [2]. Step I consists of estimating a parameter set that describes the model and step II consists of determining the relevant information from the parameter estimates. Both steps are important to the overall success of the method. Ill-conditioning at either step can adversely effect the overall performance of the method and needs to be avoided [2,3].

For example in the sinusoid problem, let us denote the parameter set by $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, and the quantity of interest are the frequencies ω_i's. An error $\Delta\theta$ in the estimate of θ results in a error $\Delta\omega_i$ in the frequency, where

$$\Delta\omega_i = \sum_{j=1}^{r} \frac{\partial\omega_i}{\partial\theta_j} \Delta\theta_j = \Delta\omega^t \Delta\Theta ,$$

and

$$\Delta\omega = \left[\frac{\partial\omega_i}{\partial\theta_1}, \frac{\partial\omega_i}{\partial\theta_2}, \ldots, \frac{\partial\omega_i}{\partial\theta_r} \right]^t ,$$

with $\Delta\Theta$ being defined similarly. Here we will use " t " to denote transpose, " $*$ " to denote complex conjugate, and " H " to denote complex conjugate transpose.

$$E\{|\Delta\omega_i|^2\} = \Delta\omega^t E\left\{\Delta\Theta\, \Delta\Theta^H\right\} \Delta\omega^* \le S\, \|R_\theta\| ,$$

where

$$S = \|\Delta\omega\|^2 = \sum_{j=1}^{r} \left|\frac{\partial\omega_i}{\partial\theta_j}\right|^2 \quad \text{and} \quad R_\theta = E\left\{\Delta\Theta\, \Delta\Theta^H\right\}$$

S represents the sensitivity of the parameter set and R_θ the estimation error covariance. In step I, one attempts to minimize $\|R_\theta\|$†. However for the overall success of the method, ill-conditioning in step II has to be avoided, i.e. S has to be small. Reduced sensitivity also reduces the demands on the estimation process. Though an optimality in both steps is desirable, in practice, this may be hard to achieve.

In this paper we consider the problem of sinusoids, damped sinusoids and then the general rational modeling (ARMA) problem. In all these cases, we concentrate on step II, the parameterization issue and attempt to determine robust parameter sets. For this purpose we focus our attention on state space parameterization and pay particular attention to balanced coordinate systems [4,5,6].

II. SINUSOID PROBLEM

For the sinusoid problem, the data is the sum of complex exponentials, i.e.

$$y(k) = \sum_{i=1}^{p} c_i\, e^{j\omega_i k}$$

Such a signal can be modeled by a state space model

$$X_{k+1} = F\, X_k \tag{1a}$$

$$y(k) = h\, X_k . \tag{1b}$$

where the eigenvalues of F are of unit amplitude and equal to $e^{j\omega_i}$ [7]. One interesting realization is when

$$F = diag(e^{j\omega_i}), \tag{2}$$

and $x(0)^t = (c_1, c_2, \ldots, c_p)$ and $h = (1,1,\ldots,1)$. The triple $(F, x(0), h)$ characterize the model and are not unique. If $(F, x(0), h)$ is one representation then $(T^{-1}FT, T^{-1}x(0), hT)$ is also a representation for any nonsingular matrix T. Here since the eigenvalues of F are of interest, a matrix whose eigenvalues are insensitive to perturbation is desirable for robustness in step II. This issue is examined in detail later. Now we present some factorizations that can be used to estimate the state space parameters.

From the state space equations, for the sinusoidal signal it can be shown that

$$y(k) = hF^k x(0). \tag{3}$$

The data Hankel matrix can be factorized into

$$\begin{bmatrix} y(0) & y(1) & y(2) & \cdot \\ y(1) & y(2) & y(3) & \cdot \\ y(2) & y(3) & y(4) & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} = \begin{bmatrix} h \\ hF \\ hF^2 \\ \cdot \\ \cdot \end{bmatrix} \left[x(0),\ Fx(0),\ F^2x(0),\ \ldots \right] = \Theta\ R \tag{4}$$

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†Unless mentioned otherwise the norm used is the 2-norm.

QUALITY OF LOW RANK APPROXIMANTS OBTAINED* USING SVD AND SOME APPLICATIONS

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ABSTRACT

In this paper, we examine the problem of evaluating the quality of low rank approximants obtained using the Singular Value Decomposition (SVD), a commonly used procedure in the recently popular Subspace Based methods. The concept of angle between subspaces is used to quantify the nature of the approximant. This measure allows one to determine when a SVD based procedures fails, and also suggests procedure to maximize the performance of these methods. The potential use of this result is demonstrated by applying it to the problem of estimating the frequencies of multiple sinusoids in noise.

I. INTRODUCTION

Singular Value Decomposition (SVD) has become a very popular tool in signal processing problems. For instance it is used widely in Least Squares problems which arise naturally in many signal processing problems. In such problems one seeks a vector g which best solves

$$A\,g = h,$$

where A is a $m \times n$ matrix, g is a $n \times 1$ vector and h is a $m \times 1$ vector. Of particular interest is the case when the rank of matrix A is p where $p < \min(m,n)$. The best minimum norm solution to this least squares problem is given by

$$g = A^{+} h,$$

where A^{+} denotes the pseudoinverse of A and is often computed using a SVD (Singular Value Decomposition) [1,2]. Normally A and h are inexact due to the presence of noise. Let the perturbed matrix and vector be denoted by B and b respectively, i.e.

$$B = A + E, \text{ and } b = h + e.$$

Usually matrix B will be full rank, i.e. rank of B is equal to r where $r = \min(m,n)$. Under these conditions an often used procedure is to replace B by a p rank approximant $\hat{B}$ before taking the pseudo inverse. An effective tool for such an approximation is provided by the Singular Value Decomposition (SVD) [1,2]. If the SVD of B is given by

$$B = U\,\Sigma V^{H} = \left[\, U_1 \mid U_2 \,\right] \left[\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & \Sigma_2 \end{array} \right] \left[\begin{array}{c} V_1^{H} \\ \hline V_2^{H} \end{array} \right].$$

where U and V are the left and right singular vectors, and Σ is a diagonal matrix containing the singular values in descending order, i.e. $\sigma_1, \sigma_2, \ldots, \sigma_r$. The approximant is obtained by retaining the first p principal singular vectors (U_1 and V_1) and principal singular values (Σ_1), i.e.

$$\hat{B} = U_1 \Sigma_1 V_1^{H},$$

and

$$\hat{B}^{+} = V_1 \Sigma_1^{-1} U_1^{H}.$$

The least square solution, denoted by x, is then determined as

$$x = \hat{B}^{+} b.$$

There are a number of examples in signal processing which can be treated in the above framework, e.g. estimating the frequencies of multiple sinusoids [3,4], estimating the direction of arrival [5,6] etc. SVD is also used extensively in approximating a given matrix by a low rank matrix and in some application the subspaces associated with the singular vectors are used for parameter estimation [12,9]. Such an application will be pursued in sec III.

Inspite of the success of SVD based procedures, the question of evaluating the estimation procedure still remains. Some important questions that still needs to be addressed are the following:

a) How good is the approximant obtained by using the SVD for estimation problems?

b) When do such SVD based approximation procedures fail to be advantageous or useful.

These are the issues that we attempt to address in this paper. Some results were first presented in [7,8,9].

II. MAIN RESULT

For evaluating the quality of the approximant, and hence the reliability of the estimation procedure, the concept of angle between subspaces is considered in detail here. The angle between two subspaces L and M is defined as

$$\| Sin\theta(L,M) \|_2 = \| (I - P_M) P_L \|_2 = \| P_M^{\perp} P_L \|_2$$

where P_M and P_L are the projection operators onto the subspaces M and L respectively, and is a natural generalization of the angle between vectors [2,10,11]. If the dimension of the subspaces are equal, then

$$\| P_M - P_L \| = \| P_M^{\perp} P_L \| = \| P_L^{\perp} P_M \|.$$

Consequently, the concept of angle between subspaces provides us a useful tool to characterize the difference between two subspaces. It is desirable to have $\| P_M - P_L \| < 1$, since this ensures that no vector in the subspace M is orthogonal to a vector in the subspace L [2,10,11].

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NUMERICAL CONSIDERATIONS IN SPECTRUM ESTIMATION †

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ABSTRACT

In this paper, we consider the problem of determining reliable model based procedures for estimating the frequencies of sinusoids from noisy measurements. This is done by analyzing the two steps involved in such a model based approach. One of the steps involves analyzing the parameter sensitivity of the model parameter set, and the other involves examining the reliability of the estimation procedure used to determine the parameter set. For this study, we examine the use of state space models for estimating the frequencies of multiple sinusoids in noise. In particular, robust parameter sets are identified, and procedures to estimate the set are presented. It is shown that the ideal parameter set involves estimating an unitary matrix, and then computing its eigenvalues to obtain an estimate of the frequencies. Reliable Singular Value Decomposition (SVD) based procedures to estimate unitary or near unitary matrices from covariance and time series data are presented.

I. INTRODUCTION

The ability of model based spectral estimation procedures to produce high resolution estimates from finite data records has attracted a great deal of attention [1-3]. A multitude of techniques have been suggested and for finite data records comparisons based on simulations have been done. In this paper, a two step procedure is used to study the numerical issues involved in such model based methods. The main motivation for such an approach is the fact that model based methods can in general be divided into two distinct steps as in fig.(1). Step I consists of estimating a parameter set that describes the model, and step II consists of determining the relevant information from the parameter estimates. Both steps are important to the overall success of the method. Ill-conditioning at either step can adversely effect the overall performance of the method and needs to be avoided. Reliable estimation (Step I) is clearly an important issue and has been given a fair amount of attention. For example, in the deterministic least squares problem reliable estimation procedures, e.g. QR approach, SVD approach, have been

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A.2.7

E3.20

AN IMPROVED TOEPLITZ APPROXIMATION METHOD[1]

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ABSTRACT

In this paper, we suggest a modification of the Toeplitz approximation method for estimating frequencies of multiple sinusoids from covariance measurements. The method constructs a state-feedback matrix following a low-rank approximation of the Toeplitz covariance matrix via singular value decomposition. Ideally, the eigenvalues of this state-feedback matrix will be on the unit circle in the complex plane, and the angles that they make with the real axis will be equal to the unknown sinusoid frequencies. The modification proposed here exploits this prior knowledge of the modulus of the eigenvalues, and guarantees that even in the presence of noise, the eigenvalues of the estimated state-feedback matrix will lie on the unit circle.

1. INTRODUCTION

The problem of retrieving multiple sinusoids (with frequencies close to each other) from perturbed time-series or covariance information is of special interest in a vast range of signal-processing applications. Very often the covariance sequence may have to be estimated from time-series data, as in Doppler processing in radar. It is not uncommon, however, to encounter applications in which the (time-series) data are not measurable while the covariance information is directly available. Such situations arise in astronomical star bearing estimation, interference spectroscopy, and some sensor array applications.

In recent years, there has been a great deal of interest in model-based sinusoid retrieval. Models convert the non-linear problem of estimating the sinusoid frequencies into a simpler problem of estimating the parameters of a linear model [1]. The second step in all model-based methods is the extraction of the desired information (the frequencies) from the estimated model parameters [2]. Both steps are important for the overall success of a model-based method. Ill-conditioning at either step can adversely affect the overall performance of the method and should be avoided. The reliability of the first step depends on the estimation procedure, and that of the second step on the sensitivity of the desired information to the model parameters [3]. A popular model for the sinusoid retrieval problem is the linear prediction model first used by Prony in 1881.

$$y(t) = \sum_{k=1}^{p} a_k y(t-k)$$

whose parameters may be reliably estimated by the method of Tufts and Kumaresan [4]. The roots of the polynomial formed from these parameters are ideally expected to be on the unit circle in the complex plane, and the angles that they make with the real axis should equal the sinusoid frequencies.

2. STATE-SPACE REPRESENTATION

It turns out that the sinusoidal model is a very special case of the general linear rational model, and that just as there are alternate parameterizations of linear systems, there also are alternate parameter sets for the sinusoidal model as well. Just as there is a state-space representation for every realization of a linear, rational system, there is also a state-space representation for every realization of the sinusoidal model. The state-space representation of the special model for sinusoidal signals (frequencies: ω_i, $i=1,2,\ldots,n$) is:

$$x(k+1) = Fx(k)$$
$$y(k) = hx(k)$$

where the order of the model p is twice the number of sinusoids, and the eigenvalues of F are of unit magnitude and equal $e^{\pm j\omega_i}$, $i=1,2,\ldots,n$. The sinusoidal signal $y(t)$ is the model's zero-input response to some non-zero initial condition $x(0)$. In fact, we have

$$y(t) = hF^t x(0), \quad t \geq 0,$$

and the covariance $r(m)$ of the sinusoidal signal satisfies

$$r(m) = hF^m P h^\dagger \quad m \geq 0 \tag{1}$$

where P is the state-covariance matrix, and the superscript † denotes the Hermitian transpose.

The linear prediction model is a canonical realization of the above, with

$$x(t) = \left[y(t-1),\ y(t-2) \cdots y(t-p) \right]^t,$$

$$F = \begin{bmatrix} a_1 & a_2 & . & . & a_p \\ 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 0 \end{bmatrix}, \quad h = \left[a_1\ a_2\ \cdots\ a_p \right].$$

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LOWERING THE THRESHOLD SNR OF SINGULAR VALUE DECOMPOSITION* BASED METHODS

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ABSTRACT

In this paper, we examine the performance of Singular Value Decomposition (SVD) based methods for estimating the frequencies of multiple sinusoids. The concept of angle between subspaces is used to derive a criteria for determining when SVD based procedures fail. The signal to noise ratio (SNR) at which a method breaks down is termed the threshold SNR of the method. It is then shown that existing SVD based methods have a higher threshold SNR than predicted by this criteria. A method that utilizes the singular vectors, and directly minimizes the angle between subspaces is developed. The method is shown to have better performance at low signal to noise ratios. The procedure lowers the threshold SNR thereby extending the range of SNR for which SVD can be used.

I. INTRODUCTION

Singular Value Decomposition (SVD) has become a very popular tool in signal processing problems. In this paper we examine its use for the problem of estimating the frequencies of multiple sinusoids in white noise. SVD has been found to be a very useful tool in obtaining reliable estimates of the frequencies. However, very little is known about certain fundamental issues relating to its use. The issues considered in this paper are:
1) When do SVD based methods fail?
2) Do the SVD based methods that exist for the sinusoid problem make the best use of the decomposition?
3) If not, are there procedures that can make the most out of using SVD?
In this paper, a criteria based on the angle between subspaces is considered to determine when the SVD based methods fail. The signal to noise ratio (SNR) at which a method breaks down is termed the threshold SNR of the method. A low threshold SNR is a desirable feature for a method. A SVD based State Space approach is considered [9,12], and is shown to have a higher threshold SNR than predicted by this criteria. The same is true for the SVD based Linear Prediction method [3,7]. A method that utilizes the singular vectors, and directly minimizes the angle between subspaces is then developed. The method is shown to have better performance at low signal to noise ratios, and has a lower threshold SNR. The results also indicate that existing SVD based methods do not make the best use of the subspaces.

II. BACKGROUND

Here we briefly describe a SVD based state space approach for the sinusoid frequency estimation problem [9,12]. For this discussion we assume that the data is noise free and address the issue of noise later. For the sinusoid problem, the data is the sum of complex exponentials, i.e.

$$x(k) = \sum_{i=1}^{p} c_i \, e^{j\omega_i k},$$

where ω_i are the frequencies and c_i are the complex amplitude with $c_i = |c_i| e^{j\phi_i}$, ϕ_i being the phase. Such a signal can be modeled by the following state space model

$$X_{k+1} = F\, X_k$$

$$x(k) = h\, X_k\,,$$

where the eigenvalues of F are of unit amplitude and equal to $e^{j\omega_i}$ [12,9]. This can be verified by considering the following useful realization.

$$F = diag(e^{j\omega_i}), \tag{1a}$$

$$X_0^t = (c_1, c_2, \ldots, c_p) \quad \text{and} \quad h = (1,1,\ldots,1). \tag{1b}$$

The triple (F, X_0, h) characterize the model and are not unique. If (F, X_0, h) is one representation then $(T^{-1}FT, T^{-1}X_0, hT)$ is also a representation for any nonsingular matrix T. This nonuniqueness of the parameter set enables one to choose a coordinate system that is less sensitive to perturbations [13,9,12]. The state space parameters can be estimated by using the factorization of matrix D given below.

$$D = \begin{bmatrix} x(1) & x(2) & \cdots & x(N-L+1) & x^*(N) & x^*(N-1) & \cdots & x^*(L) \\ x(2) & x(3) & \cdots & x(N-L+2) & x^*(N-1) & x^*(N-2) & \cdots & x^*(L-1) \\ \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \\ \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \\ x(L) & x(L+1) & \cdots & x(N) & x^*(N-L+1) & x^*(N-L) & \cdots & x^*(1) \end{bmatrix}$$

$$= \begin{bmatrix} h \\ hF \\ \cdot \\ \cdot \\ hF^{L-1} \end{bmatrix} \left[X_1, FX_1, \cdots, F^{N-L}X_1, Z_n, FZ_n, \cdots, F^{N-L}Z_n \right]$$

$$= \Theta\, \Phi. \tag{2}$$

where Θ is shown below.

$$\Theta = \begin{bmatrix} h \\ hF \\ hF^2 \\ \cdot \\ \cdot \\ hF^{L-1} \end{bmatrix}.$$

The matrix D arises naturally in a forward and backward approach [3]. The matrix F can be estimated from Θ in the following manner:

$$\Theta_1 F = \Theta_2 \tag{3}$$

where

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Perturbation Analysis of an SVD-Based Linear Prediction Method for Estimating the Frequencies of Multiple Sinusoids

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Abstract—Model-based spectral estimation techniques consist of essentially two steps. The first step is the estimation of a parameter set from the data, and the second step consists of extracting the relevant information from the parameter set. A numerical analysis of the overall procedure can be performed by conducting a perturbation analysis of the two steps separately. We demonstrate this by studying the linear prediction approach for estimating the frequencies of sinusoids in white noise. It is shown that in the first step, the continuity of the generalized inverse and the concept of angle between subspaces play an important role. The continuity concept helps explain the need for a low rank approximation, and the quality of the approximant is appraised by using the notion of angle between subspaces. For the second step, the sensitivity of the zeros of the predictor polynomial becomes an important consideration and is examined. It is shown that increasing the order of the predictor polynomial and computing the minimum norm solution provides a mechanism to reduce parameter sensitivity.

I. Introduction

In recent years, there has been a flurry of activity in the applications of models to spectral estimation, and they show great promise [1]. There is a multitude of techniques with asymptotically similar behavior, and so their performance for finite data records becomes an important consideration. Comparisons based on simulations have been used to evaluate the methods. In addition to the statistical behavior, a numerical examination is useful to understand their reliability/performance in a finite-precision environment, i.e., when fixed-point or floating-point arithmetic is used. In this paper we consider this issue.

To begin the analysis, we note that the model-based methods can, in general, be divided into two distinct steps as in Fig. 1. Step I consists of estimating a parameter set that describes the model, and Step II consists of determining the relevant information from the parameter estimates. Both steps are important to the overall success of the method. Ill-conditioning at either step can adversely effect the overall performance of the method and needs to be avoided. Reliable estimation (Step I) is clearly an important issue and has been given a fair amount of attention. For example, in the deterministic least squares problem, reliable estimation procedures, e.g., the QR and SVD approaches, have been developed [2], [3]. However, for the spectral estimation procedure, in addition to the first step, the second step is also an important factor. If the mapping from the parameter space to the spectral domain is ill-conditioned, then improving Step I is less effective. On the other hand, a reliable parameter set may not always be easy to estimate. So a compromise is necessary and a proper combination of the two steps is important for success. An evaluation of the two steps separately provides a procedure for evaluating and comparing methods, and results in interesting insights.

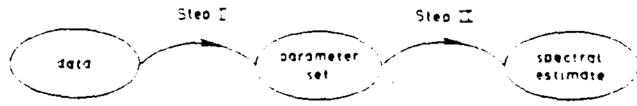

Fig. 1. The two steps in a model-based spectrum estimation method.

Here we consider the problem of estimating frequencies of sinusoids in white noise. For the sinusoid problem, let us denote the parameter set by $\theta = [\theta_1, \theta_2, \cdots, \theta_r]$ and the quantity of interest are the frequencies ω_i's. An error $\Delta\theta$ in the estimate of θ results in a error $\Delta\omega_i$ in the frequency where

$$\Delta\omega_i = \sum_{j=1}^{r} \frac{\partial\omega_i}{\partial\theta_j} \Delta\theta_j = \nabla\omega_i^t \Delta\theta$$

where

$$\nabla\omega_i = \left[\frac{\partial\omega_i}{\partial\theta_1}, \frac{\partial\omega_i}{\partial\theta_2}, \cdots, \frac{\partial\omega_i}{\partial\theta_r}\right]^t$$

and

$$\Delta\theta = [\Delta\theta_1, \Delta\theta_2, \cdots, \Delta\theta_r]^t.$$

The error $\Delta\theta$ is a result of the additive noise in the data, and is also due to finite precision arithmetic.

$$E|\Delta\omega_i|^2 = \nabla\omega_i^t E[\Delta\theta\Delta\theta^t]\nabla\omega_i \le S_\omega \|R_{\theta\theta}\|^1$$

where

$$S_\omega = \|\nabla\omega_i\|^2 = \sum_{j=1}^{r} \left|\frac{\partial\omega_i}{\partial\theta_j}\right|^2$$

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[1]Unless mentioned otherwise, the norm referred to in this paper is the 2 norm [2], [3].

PERFORMANCE ANALYSIS OF SUBSPACE BASED METHODS*

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ABSTRACT

In this paper, we analyze the performance of two subspace based methods, ESPRIT, and the Toeplitz Approximation Method (TAM), for estimating the direction of arrival (DOA) of plane waves in white noise. In the case of a linear equispaced sensor array, it is shown that ESPRIT, like TAM, can be treated via a state space formulation leading to interesting insights. The state transition matrix estimated by ESPRIT is shown to be related to that obtained by TAM by a diagonal similarity transformation. The performance of these methods is also analyzed. In particular, asymptotic results for the mean squared error in the estimates of the direction of arrival are derived for each of the above mentioned methods. Simple closed form expressions for the one source case are derived, and compared with the corresponding expressions for the Minimum-Norm method and MUSIC. Computer simulations are provided to substantiate the analysis.

INTRODUCTION

Subspace based methods have been developed and studied by a number of researchers. In this paper we analyze the performance of the subspace based methods used in estimating the Direction of Arrival (DOA) of plane waves in noise. Due to length considerations, in this paper, we present only a detailed examination of ESPRIT [1], and the Toeplitz Approximation Method (TAM) [3]. The approach can be extended to analyze the Minimum-Norm method [2,10] and MUSIC [4,5].

MUSIC was the first method that showed the benefits of using a subspace based approach [4]. Some theoretical results comparing MUSIC, and the Minimum-Norm method can be found in [6,7] wherein a characterization of the methods was done by examining the null spectrum. Some comparisons of MUSIC with ESPRIT can be found in [9]. ESPRIT is, like MUSIC, a general approach, and was developed to overcome some of the computation and prior information requirements of MUSIC [1,8,9]. It is conceptually different in that it calls for an array of identical doublets, i.e. requires the array to possess a displacement invariance. In return, it does not need detailed information about the array geometry and element characteristics. Here we only consider its performance in the context of a linear equispaced sensor array. The use of the Toeplitz Approximation Method (TAM) for DOA estimation was first suggested in [3]. TAM is a method, like the Minimum-Norm method, that was first developed for the sinusoid frequency estimation problem [3,11]. A key feature of the method is that it is based on a state space model. Such modeling has benefits that were discussed in [11,12,13], and will become clearer as the discussion progresses. In fact it will be shown that ESPRIT can also be described under this formulation leading to insights into its performance.

The organization of the paper is as follows. A State Space formulation is used to describe ESPRIT, and establish the relationship between TAM and ESPRIT. Asymptotical results for the mean squared error in the estimates of the DOA in each of the two cases are derived and compared. The results are specialized for the one source case, and compared with the corresponding expressions for the Minimum-Norm method and MUSIC. Simulation results are presented and they support the analysis.

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PROBLEM FORMULATION

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors is considered in this paper. For the kth observation period (snapshot), the spatial samples of the signal plus noise are given by

$$Y_k^T = \left[y_1^{(k)}, y_2^{(k)}, \cdots, y_L^{(k)} \right] = \left[\sum_{i=1}^{M} p_i^{(k)}, \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] + N_k^T, \quad (1)$$

where $\omega_i = \frac{2\pi d}{\lambda}\sin\theta_i$, d being the separation between sensors, λ the wavelength of the incident signal, and θ_i the direction of arrival. The subspace based methods estimate the signal zeros $z_i = e^{j\omega_i}$, $i = 1,\ldots,M$, from which the signal frequencies ω_i's and then the DOA's θ_i are determined. As in [6,7], it is assumed that N_k is a sequence of independent, mean zero Gaussian random vectors, i.e $E[N_k N_l^H] = \sigma_n^2 I \delta_{kl}$. The noise is assumed independent of the complex signal amplitudes $p_i^{(k)}$ which are also modeled as being jointly Gaussian. The covariance matrix P of the amplitudes whose elements are P_{ij}, where $P_{ij} = E[p_i^{(k)} p_j^{*(k)}]$, is assumed to be of rank M and has distinct eigenvalues.

Now we make some comments regarding the notation and conventions used throughout the the paper. 'T' is used to denote transpose, * to denote complex conjugate, 'H' to denote complex conjugate transpose and + to denote the Moore-Penrose pseudo-inverse. Also $E[.]$ and the superscript '$-$' will be used interchangeably to denote the expectation operator. In this paper, $\hat{}$ is used to denote estimates, and subscript s to denote parameters associated with the signal alone. Also if y is a vector of length L, then y' and y'' denote vectors of length $L-1$ formed from the first and last $L-1$ elements of y, i.e. if $y = [y_1, y_2, \ldots, y_{L-1}, y_L]^T$, then $y' = [y_1, y_2, \ldots, y_{L-1}]^T$, and $y'' = [y_2, \ldots, y_{L-1}, y_L]^T$. For compactness of representation, we define two $(L-1)xL$ matrices W' and W''

$$W' = \left[I_{(L-1)x(L-1)}, 0 \right] \quad W'' = \left[0, I_{(L-1)x(L-1)} \right]. \quad (2)$$

Note that $y' = W'y$ and $y'' = W''y$. For matrices we have the following convention. If B is a LxM matrix, then B_1 and B_2 are $(L-1)xM$ matrices containing the first and last $L-1$ rows of B, i.e.

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_{L-1} \\ b_L \end{bmatrix} = \begin{bmatrix} b_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ b_L \end{bmatrix}. \quad (3)$$

Also Δx is used to denote the error in quantity x, where x may be a scalar, vector or matrix.

The covariance matrix of the observation vector Y_k has an eigendecomposition and can be written as

$$R = \overline{Y_k Y_k^H} = \sum_{i=1}^{L} \lambda_i S_i S_i^H = E \Lambda E^H = E_s \Lambda_s E_s^H + \sigma_n^2 I. \quad (4)$$

PERFORMANCE ANALYSIS OF ROOT-MUSIC *

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ABSTRACT

In this paper, we analyze the performance of Root-Music, a variation of the popular MUSIC algorithm, for estimating the direction of arrival (DOA) of plane waves in white noise in the case of a linear equispaced sensor array. The performance of the method is analyzed by examining the perturbation in the roots of the polynomial formed in the intermediate step of Root-Music. In particular, aysmptotic results for the mean squared error in the estimates of the direction of arrival are derived. Simple closed form expressions are derived for the one and two source case to get further insight. Computer simulations are provided to substantiate the analysis. An important outcome of this analysis is an explanation as to why Root-Music is superior to the popular MUSIC algorithm where one examine the peaks of the spatial spectrum.

I. INTRODUCTION

Eigen-Decomposition based methods have recently been extensively used in estimating the Direction of Arrival (DOA) of plane waves in noise [1,6]. These methods, often referred to as subspace based methods, have been shown to perform very well and are capable of resolving closely spaced sources. MUSIC was the first method that showed the benefits of using a subspace based approach [1]. The MUSIC algorithm computes a spatial spectrum from the noise subspace, and determines the DOA's from the dominant peaks in the spectrum. Another popular variation of MUSIC is Root-Music [2]. Root-Music, as described in more detail later, is similar to MUSIC in many respects except that the DOA's are determined from the roots of a polynomial formed from the noise subspace. Though MUSIC is applicable to general known array configurations. Root-Music is mainly suitable in the context of a linear equispaced sensor array [2]. Some theoretical results comparing MUSIC, and the Minimum-Norm method can be found in [3,4] wherein a characterization of the methods was done by examining the null spectrum. Our work examines Root-Music, and characterizes the mean squared error in the estimates of the DOA's directly. The analysis provides insight into why Root-Music is superior to the popular MUSIC algorithm where one examines the peaks in the spatial spectrum.

II. PROBLEM FORMULATION

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors is considered in this paper. For the ***kth*** observation period (snapshot), the spatial samples of the signal plus noise are given by

$$Y_k^T = \left[\sum_{i=1}^{M} p_i^{(k)} , \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i} , \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] + N_k^T, \quad (1)$$

where $\omega_i = \frac{2\pi d}{\lambda} \sin\theta_i$, d being the separation between sensors, λ the wavelength of the incident signal, and θ_i the direction of arrival. Root-Music estimates $z_i = e^{j\omega_i}$, $i=1,...M$, the signal zeros, from which ω_i's, the signal frequencies, and then the DOA's θ_i are determined. As in [3,4], the noise vector N_k is assumed to be a zero mean, complex white Gaussian random vector, i.e. $\overline{N_k N_l^H} = \sigma_n^2 I \, \delta_{kl}$. The noise is assumed independent of the complex signal amplitudes $p_i^{(k)}$ which are also modeled as being jointly Gaussian. The covariance matrix P of the amplitudes whose elements are P_{ij}, where $P_{ij} = E[p_i^{(k)} p_j^{*(k)}]$, is assumed to be of rank M and has distinct eigenvalues.

In this paper, 'T' is used to denote transpose, * to denote complex conjugate, and H to denote complex conjugate transpose. Also $E[.]$ and the superscript "--" will be used interchangeably to denote the expectation operator. In this paper, ˆ is used to denote estimates, and subscript s and n to denote parameters associated with the signal and noise respectively.

The covariance matrix of the observation vector Y_k has an eigendecomposition and can be written as

$$R = \overline{Y_k Y_k^H} = \sum_{l=1}^{L} \lambda_l S_l S_l^H = E \Lambda E^H = E_s \Lambda_s E_s^H + \sigma_n^2 I. \quad (2)$$

where

$$E = [S_1, S_2, \cdots, S_L], \quad E_s = [S_1, S_2, ..., S_M] \quad (3a)$$

$$\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_L), \text{ and}$$

$$\Lambda_s = diag(\lambda_1^s, \lambda_2^s, \cdots, \lambda_M^s). \quad (3b)$$

Also

$$\lambda_1 = \lambda_1^s + \sigma_n^2 > \lambda_2 = \lambda_2^s + \sigma_n^2 > .. > \lambda_M = \lambda_M^s + \sigma_n^2$$
$$> \lambda_{M+1} = \cdots = \lambda_L = \sigma_n^2.$$

λ_i are the eigenvalues of R, and S_i the corresponding orthonormal eigenvectors. The subspace based methods estimate the DOA's from either the subspace E_s, i.e. the space spanned by the eigenvectors corresponding to the M dominant eigenvalues, termed the signal subspace, or its orthogonal complement E_N, where $E_N = [S_{M+1}, \cdots, S_L]$, termed the noise subspace. P_S and P_N are used to denote the orthogonal projection operators onto the signal and noise subspace respectively.

This paper considers the effect of using an estimated covariance matrix. Usually an estimate of the covariance matrix is obtained by (time) averaging N independent snapshots, i.e.

$$\hat{R} = \frac{1}{N} \sum_{k=1}^{N} Y_k Y_k^H = \hat{E} \hat{\Lambda} \hat{E}^H,$$

where

$$\hat{E} = [\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_L], \quad \hat{\Lambda} = diag(\hat{\lambda}_k).$$

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C.12

STATE SPACE METHODS FOR DOA ESTIMATION*

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ABSTRACT

In this paper, we examine state space methods for estimating the direction of arrival of plane waves from a linear equispaced sensor array. From a computational point of view, the approach only involves matrix operations and is suitable for systolic/wavefront implementation.

I. INTRODUCTION

Development of signal processing algorithms for implementation on Systolic/Wavefront architectures have been of interest in the past few years [1,2,3]. In this paper, we examine methods for estimating the direction of arrival (DOA) of plane waves using a linear equispaced sensor array with a view to implement them on systolic/wavefront arrays. In the context of DOA estimation, subspace based methods have recently received a great deal of attention. Examples of such methods are MUSIC [4], ESPRIT [5], the Toeplitz Approximation Method (TAM) [6] etc. In particular MUSIC has received a great deal of attention. The various steps of a MUSIC algorithm are summarized in Fig. 1. Note that there are efficient systolic architectures to perform the first two steps, namely covariance computation and eigendecomposition. However the process of computing the null spectrum $D(\omega)$ and locating the peaks in its inverse is computationally expensive and not very amenable to systolic processing. In this paper, we examine state space based methods. State space methods for these problems were first suggested in [6,7]. The computational steps in the method are based solely on matrix operations making it an attractive candidate for systolic implementations. Note that ESPRIT for a linear equispaced array can be treated via this formalism. Because of its applicability to more general array configurations, the ideas presented here have wider applicability.

II. STATE SPACE APPROACH

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors using state space methods is considered in this paper. For the kth observation period (snapshot), the spatial samples of the signal plus noise are given by

$$Y_k^T = \left[y_1^{(k)}, y_2^{(k)}, \cdots, y_L^{(k)} \right]$$
$$= \left[\sum_{i=1}^{M} p_i^{(k)}, \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \cdots, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] + N_k^T, \tag{1}$$

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C-11

ROUNDOFF NOISE IN FLOATING POINT STATE SPACE DIGITAL FILTERS*

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ABSTRACT

In this paper, the roundoff noise properties of floating point state-space digital filters is examined. It is shown that the roundoff noise behavior of these filters is related to their coefficient sensitivity. Expression for the variance of the output roundoff noise is derived when the input to the filter is a zero mean wide sense stationary random process. These expressions, as expected, depend on the input signal statistics along with the filters parameters, in particular the controllability and observability grammians.

I. INTRODUCTION

Floating point arithmetic is widely used in signal processing computations. More recently, with the increasing availability of floating point capability in signal processing chips, insight into algorithms employing floating point arithmetic is of interest. This paper considers the problem of digital filter implementation, and examines the effect of floating point arithmetic on them. In particular floating point state-space digital filters are analyzed. The problem of using fixed point arithmetic and design of minimal roundoff filters is well known [1,2]. In contrast, little is known about roundoff noise in filters employing floating point arithmetic. Effect of floating point arithmetic on digital filters has been studied in the context of direct, cascade and parallel forms [3,4], and partially for state space digital filters [5]. Here a detailed treatment of the effects of using floating point arithmetic in state-space digital filters is presented.

II. FLOATING POINT ARITHMETIC

Throughout this paper it will be assumed that floating point numbers are stored in the form $(sign)\, u \cdot 2^{v}$, where u and v have a fixed number of bits. Also it will be assumed that rounding is used in all operations. Truncation can be dealt with in a similar manner. The notation $fl(.)$ will be used to denote the machine number resulting from floating point operation. In floating point operations there are errors in both additions and multiplications [3,4], i.e.

$$fl(x+y) = (x+y)(1+\varepsilon),$$

and

$$fl(xy) = xy(1+\delta),$$

and $-2^{-q} \le \varepsilon, \delta \le 2^{-q}$, q being the number of bits used to represent the mantissa. The error variables ε and δ are assumed to be random variables uniformly distributed between -2^{-q} and 2^{-q}. The error variables have zero mean and variance $\sigma_\varepsilon^2 = \frac{2^{-2q}}{3}$. The error in the case of addition is $\varepsilon(x+y)$ and in the case of multiplication is δxy. A significant difference between fixed point and floating point arithmetic is that the error caused by rounding in floating point arithmetic is dependent on the signal. This dependence of the rounding error on the signal makes the analysis less tractable. To carry out the analysis some careful manipulations are necessary. We now consider in detail the floating point computation of the inner product, a computation often used in a state-space filter. The inner product of the vector $\underline{a} = (a_1, a_2, a_3)^t$ and $\underline{x} = (x_1, x_2, x_3)^t$ is given by

$$y = \underline{a}^t \underline{x} = a_1 x_1 + a_2 x_2 + a_3 x_3.$$

As will be clear from the discussion below, the error due to finite precision floating point arithmetic will depend on the order of the operations.

$$y_1 = fl(fl(a_1x_1) + fl\{fl(a_2x_2) + fl(a_3x_3)\}) \tag{1a}$$

$$= (a_1x_1(1+\delta_1) + [a_2x_2(1+\delta_2) + a_3x_3(1+\delta_3)](1+\varepsilon_1))(1+\varepsilon_2)$$

$$= a_1(1+\Delta_1)x_1 + a_2(1+\Delta_2)x_2 + a_3(1+\Delta_3)x_3.$$

$$= (a_1+\Delta a_1)x_1 + (a_2+\Delta a_2)x_2 + (a_3+\Delta a_3)x_3$$

$$= (\underline{a} + \Delta\underline{a})^t \underline{x}. \tag{1b}$$

where $\Delta a_i = \Delta_i a_i$ and $\Delta_1 = \delta_1 + \varepsilon_2$ and $\Delta_2 = \delta_2 + \varepsilon_1 + \varepsilon_2$ and $\Delta_3 = \delta_3 + \varepsilon_1 + \varepsilon_2$. Such a first order approximation is reasonably accurate since the errors will be assumed to be i.i.d. random variables. Such approximations will be made through this paper. Note that[†] $\overline{(\Delta a_1)^2} = 2a_1^2\sigma_\varepsilon^2$, $\overline{(\Delta a_2)^2} = 3a_2^2\sigma_\varepsilon^2$ and $\overline{(\Delta a_3)^2} = 3a_3^2\sigma_\varepsilon^2$. Also $\overline{\Delta a_1 \Delta a_2} = a_1 a_2 \sigma_\varepsilon^2$, $\overline{\Delta a_1 \Delta a_3} = a_1 a_3 \sigma_\varepsilon^2$ and $\overline{\Delta a_2 \Delta a_3} = 2a_2 a_3 \sigma_\varepsilon^2$. Note that the resulting errors are correlated. If the operations of the inner product were done in a different order, then the resultant errors have different characteristics, i.e.

$$y_2 = fl(fl(fl(a_1x_1) + fl(a_2x_2)) + fl(a_3x_3))$$

$$= (a_1 + \Delta' a_1)x_1 + (a_2 + \Delta' a_2)x_2 + (a_3 + \Delta' a_3)x_3.$$

The errors $\Delta' a_i$ are correlated and have different statistics. An important observation in this context is that the inner product obtained using floating point arithmetic is the exact inner product of a perturbed vector $\underline{a}$, i.e. $\underline{a} + \Delta\underline{a}$, and $\underline{x}$. In this paper we will show that considerable insight can be obtained by using the above simple interpretation of the floating point inner product computation process. Note that the statistics of $\Delta\underline{a}$ depend on the order in which the inner product is computed. We now apply this model to the problem of state space digital filters.

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† The overbar denotes the expectation operator

U4.15

STATISTICAL PERFORMANCE ANALYSIS OF THE MINIMUM-NORM METHOD*

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ABSTRACT

In this paper, we analyze the performance of the Minimum-Norm method for estimating the direction of arrival (DOA) of plane waves in white noise in the case of a linear equispaced sensor array. The performance of the method is analyzed by examining the perturbation in the roots of the polynomial formed in the intermediate step of the Minimum-Norm method. In particular, aysmptotic results for the mean squared error in the estimates of the direction of arrival are derived. Simple closed form expressions are derived for the one and two source case to get further insight. Computer simulations are provided to substantiate the analysis. The results obtained are compared to those obtained for Root-Music and it is shown that the relative performance of the two methods is directly dependent on the ratio of their parameter sensitivities.

I. INTRODUCTION

Eigen-Decomposition based methods have recently been extensively used in estimating the Direction of Arrival (DOA) of plane waves in noise. These methods, often referred to as subspace based methods, have been shown to perform very well, and are capable of resolving closely spaced sources. In recent years a statistical evaluation of these methods has been conducted by a number of researchers [2-4,9]. Our work examines the Minimum-Norm method [1], and characterizes the mean squared error in the estimates of the DOA's. Motivated by our observations regarding Root-MUSIC [4], we examine the error in the roots of the polynomial formed in the intermediate step of the Minimum-Norm method. The results obtained are compared with those for Root-Music leading to interesting insights.

II. PROBLEM FORMULATION

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors is considered in this paper. For the kth observation period (snapshot), the spatial samples of the signal plus noise are given by

$$Y_k^T = \left[\sum_{i=1}^{M} p_i^{(k)}, \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \cdots, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] + N_k^T, \quad (1)$$

where $\omega_i = \frac{2\pi d}{\lambda} \sin\theta_i$, d being the separation between sensors, λ the wavelength of the incident signal, and θ_i the direction of arrival. As in [2,3], the noise vector N_k is assumed to be a zero mean, complex white Gaussian random vector, i.e. $\overline{N_k N_l^H} = \sigma_n^2 I \delta_{kl}$. The noise is assumed independent of the complex signal amplitudes $p_i^{(k)}$ which are also modeled as being jointly Gaussian. The covariance matrix P of the amplitudes whose elements are P_{ij}, where $P_{ij} = \overline{[p_i^{(k)} p_j^{*(k)}]}$, is assumed to be of rank M and has distinct eigenvalues. In this paper, the overbar ("--") will be used to denote the expectation operator.

Subspace based methods estimate $z_i = e^{j\omega_i}$, $i=1,\ldots,M$, the signal zeros, from which ω_i's, the signal frequencies, and then the DOA's θ_i are determined. They utilize the eigen-decomposition of the covariance matrix of the observation vector Y_k, i.e.

$$R = \overline{Y_k Y_k^H} = \sum_{l=1}^{L} \lambda_l S_l S_l^H = E \Lambda E^H = E_s \Lambda_s E_s^H + \sigma_n^2 I, \quad (2)$$

where

$$E = [S_1, S_2, \cdots, S_L], \quad E_s = [S_1, S_2 \ldots S_M] \quad (3a)$$

$\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_L)$, and

$$\Lambda_s = diag(\lambda_1^s, \lambda_2^s, \cdots, \lambda_M^s). \quad (3b)$$

Also

$$\lambda_1 = \lambda_1^s + \sigma_n^2 > \lambda_2 = \lambda_2^s + \sigma_n^2 > \ldots > \lambda_M = \lambda_M^s + \sigma_n^2$$
$$> \lambda_{M+1} = \cdots = \lambda_L = \sigma_n^2.$$

λ_i are the eigenvalues of R, and S_i the corresponding orthonormal eigenvectors. This paper considers the effect of using an estimated covariance matrix. Usually an estimate of the covariance matrix is obtained by (time) averaging N independent snapshots, i.e.

$$\hat{R} = \frac{1}{N} \sum_{k=1}^{N} Y_k Y_k^H = \hat{E} \hat{\Lambda} \hat{E}^H,$$

where

$$\hat{E} = [\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_L], \quad \hat{\Lambda} = diag(\hat{\lambda}_k).$$

Let $\hat{S}_k = S_k + \eta_k$ and $\hat{\lambda}_k = \lambda_k + \beta_k$. The analysis makes use of the asymptotic properties of the errors η_k and β_k. Asymptotically the errors are jointly Gaussian random variables with the errors in the eigenvectors being independent of the errors in the eigenvalues [5]. It has been shown in [5] that for the errors corresponding to the M signal eigenvectors,

$$\overline{\eta_k \eta_l^H} = \frac{\lambda_k}{N} \sum_{\substack{r=1 \\ r\neq k}}^{L} \frac{\lambda_r}{(\lambda_k - \lambda_r)^2} S_r S_r^H \delta_{kl} + o(N^{-1}), \quad (4)$$

and

$$\overline{\eta_k \eta_l^T} = -\frac{\lambda_l \lambda_k}{N(\lambda_k - \lambda_l)^2} S_l S_k^T (1 - \delta_{kl}) + o(N^{-1}), \quad (5)$$

where δ_{kl} is the kronecker delta. In [2,3], it was shown that

$$\overline{\eta_k} = -\frac{\lambda_k}{2N} \sum_{\substack{l=1 \\ l\neq k}}^{L} \frac{\lambda_l}{(\lambda_k - \lambda_l)^2} S_k = a_k S_k + o(N^{-1}). \quad (6)$$

III. MINIMUM-NORM METHOD

In the Minimum-Norm method [1], the DOA's are found by locating the peaks of $S(\omega)$, where

$$S(\omega) = \frac{1}{D(\omega)},$$

and $D(\omega)$, termed the null spectrum [2,3], is defined as

$$D(\omega) = |[1, g^H] V(\omega)|^2,$$

with $V(\omega) = \frac{1}{\sqrt{L}} [1, e^{j\omega}, \cdots, e^{j(L-1)\omega}]^T$. $[1, g^T]^T$ is a

+ denotes the pseudoinverse, T denotes transpose, * denotes complex conjugate, H denotes complex conjugate transpose. Also ^ is used to denote estimates and subscript s and n denote parameters associated with signal and noise respectively.

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SENSITIVITY CONSIDERATIONS IN STATE SPACE MODEL †
BASED HARMONIC RETRIEVAL METHODS

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ABSTRACT

In this paper, state space models are considered for estimating the frequencies of multiple sinsusoids, and robust coordinate systems for frequency estimation are identified. It is shown that the ideal parameter set involves estimating an unitary state transition matrix, and then computing its eigenvalues to obtain an estimate of the frequencies. Procedures to estimate such matrices from covariance and time series data are examined. It is shown that two state space methods, the Toeplitz Approximation Method (TAM) and the Direct Data Approximation (DDA), estimate robust state transition matrices.

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PERFORMANCE ANALYSIS OF ESPRIT AND TAM IN DETERMINING* THE DIRECTION OF ARRIVAL OF PLANE WAVES IN NOISE

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ABSTRACT

In this paper, two subspace based methods, ESPRIT and the Toeplitz Approximation Method (TAM), for estimating the direction of arrival (DOA) of plane waves in white noise in the case of a linear equispaced sensor array are evaluated. It is shown that the least squares version of ESPRIT and TAM result in the same estimate, and are statistically equivalent. It is shown that asymptotically, the estimates obtained using Least Squares ESPRIT and Total Least Squares ESPRIT have the same mean squared error. Expressions for the asymptotic mean squared error in the estimates of the direction of arrival are derived for the methods. Simple closed form expressions are derived for the one and two source case to get further insight. Computer simulations are provided to substantiate the analysis.

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C. 8

PERFORMANCE ANALYSIS OF ROOT-MUSIC *

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ABSTRACT

In this paper, we analyze the performance of Root-Music, a variation of the popular MUSIC algorithm, for estimating the direction of arrival (DOA) of plane waves in white noise in the case of a linear equispaced sensor array. The performance of the method is analyzed by examining the perturbation in the roots of the polynomial formed in the intermediate step of Root-Music. In particular, aysmptotic results for the mean squared error in the estimates of the direction of arrival are derived. Simple closed form expressions are derived for the one and two source case to get further insight. Computer simulations are provided to substantiate the analysis. An important outcome of this analysis is an explanation as to why Root-Music is superior to the popular MUSIC algorithm where one examines the peaks of the spatial spectrum.

EDICS CATEGORY : 5.2.2

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ROUNDOFF NOISE IN FLOATING POINT STATE SPACE DIGITAL FILTERS*

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ABSTRACT

In this paper, the roundoff noise properties of floating point state-space digital filters is examined. It is shown that the roundoff noise behavior of these filters is related to their coefficient sensitivity. An exact expression for the variance of the output roundoff noise is derived when the input to the filter is a zero mean wide sense stationary random process. Surprisingly, these expressions are tractable and depend on the input signal statistics along with the filter parameters, in particular the controllability and observability grammians. For the case where double precision accumulation is used, it is shown that the optimal filters are similar to those obtained when using fixed point arithmetic. Also the roundoff noise of (single precision) second order filter structures is studied in detail.

EDICS : 4.2, 4.6

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Statistical performance analysis of the minimum-norm method

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Indexing terms: Signal processing, Radionavigation, Antenna arrays, Errors and error analysis

Abstract: The performance of the minimum-norm method, both root and spectral forms, for estimating the direction of arrival (DOA) of plane waves in white noise is analysed for the case of a linear equispaced sensor array. In particular, asymptotic results for the mean squared error in the estimates of the direction of arrival are derived. Simple closed-form expressions are derived for the one and two source case to get further insight. Computer simulations are provided to substantiate the analysis. The results obtained are compared to those obtained for root-MUSIC and it is shown that the relative performance of the two methods is directly dependent on the ratio of their parameter sensitivities.

1 Introduction

In this paper, we analyse the statistical performance of an eigendecomposition based method, the minimum-norm method [1]. Eigendecomposition-based methods have recently been extensively used in estimating the direction of arrival (DOA) of plane waves in noise. These methods, often referred to as subspace-based methods, have been shown to perform very well, and are capable of resolving closely spaced sources. In recent years a statistical evaluation of these methods has been conducted by a number of researchers [2–6]. For instance, some theoretical results comparing MUSIC, and the minimum-norm method can be found in References 2 and 3, wherein a characterisation of the methods was done by examining the null spectrum. More recent work on the analysis of MUSIC can be found in References 4, 5 and 6. Some comparisons of MUSIC with ESPRIT, as well as root-MUSIC with ESPRIT, based on computer simulations, can be found in References 7 and 8. Our work examines the minimum-norm method, and characterises the mean squared error in the estimates of the DOAs. Motivated by our observations regarding MUSIC [6], we examine the error in the roots of the polynomial formed in the intermediate step of the minimum-norm method. The results obtained are compared with those for root-MUSIC, leading to interesting insights. It is shown that the relative performance of the two methods is directly dependent on the ratio of the parameter sensitivities. It is shown that this error is also related to the original minimum-norm method, in which one determines the direction of arrival from the peaks of a spectrum [1]. In addition, the analysis gives an insight into the difference between procedures that employ a root-finding approach and those that examine peaks in the spectrum.

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2 Problem formulation

The problem of estimating the direction of arrival of M incoherent plane waves incident on a linear equispaced array of L sensors is considered in this paper. For the kth observation period (snapshot), the spatial samples of the signal plus noise are given by

$$Y_k^T = [y_1^{(k)}, y_2^{(k)}, \ldots, y_L^{(k)}] = \left[\sum_{i=1}^{M} p_i^{(k)}, \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \ldots, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i}\right] + N_k^T \quad (1)$$

where

$$\omega_i = \frac{2\pi d}{\lambda} \sin \theta_i$$

d being the separation between sensors, λ the wavelength of the incident signal, and θ_i the direction of arrival. Subspace-based methods estimate the signal zeros $z_i = e^{j\omega_i}$, $i = 1, \ldots, M$, from which the signal frequencies ω_i and then the DOAs θ_i are determined. As in References 2 to 6, the noise vector N_k is assumed to be a zero mean, complex white Gaussian random vector, i.e. $\overline{N_k N_l^H} = \sigma_n^2 I \delta_{kl}$. The noise is assumed to be independent of the complex signal amplitudes $p_i^{(k)}$ which are also modelled as being jointly Gaussian. The covariance matrix P of the amplitudes whose elements are P_{ij}, where $P_{ij} = [\overline{p_i^{(k)} p_j^{*(k)}}]$, is assumed to be of rank M and has distinct eigenvalues.

The overbar '—' will be used to denote the expectation operator in this paper. Also, T is used to denote transpose, * to denote complex conjugate, H to denote complex conjugate transpose, and + to denote the Moore-Penrose pseudo-inverse. Further, ^ is used to denote estimates and subscripts s and n to denote parameters associated with the signal and noise respectively. Δx is used to denote the error in parameter x, which may be a scalar, vector or matrix.

The covariance matrix of the observation vector Y_k is given by

$$R = \overline{Y_k Y_k^H} = V_s P V_s^H + \sigma_n^2 I \quad (2a)$$

where

$$V_s = \begin{bmatrix} 1 & 1 & \cdot & 1 \\ e^{j\omega_1} & e^{j\omega_2} & \cdot & e^{j\omega_M} \\ \vdots & \vdots & & \vdots \\ e^{j(L-1)\omega_1} & e^{j(L-1)\omega_2} & \cdot & e^{j(L-1)\omega_M} \end{bmatrix} \quad (2b)$$

RELATIONSHIP BETWEEN MATRIX PENCIL AND STATE SPACE[+] BASED HARMONIC RETRIEVAL METHODS

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ABSTRACT

In this paper, we study the relationship between a state space approach and the matrix pencil method for the problem of estimating the parameters of damped and undamped sinusoids. It is shown that the methods are very similar in the exact data case, and have minor difference in implementation in the noisy data case.

EDICS : 5.1.2

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EFFECT OF SPATIAL SMOOTHING ON THE PERFORMANCE OF* MUSIC AND THE MINIMUM-NORM METHOD

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ABSTRACT

In this paper, we analyze the effect of using a spatially smoothed forward-backward covariance matrix on the performance of Root-Music and the root Minimum-Norm method for estimating the direction of arrival of plane waves in white noise in the case of a linear equispaced sensor array. In particular, aysmptotic results for the mean squared error in the estimates of the signal zeros and the direction of arrival are derived. Simple approximations are made to obtain insight into the performance of the methods. It is shown that the use of a forward-backward covariance estimate and spatial smoothing is far more beneficial to the Minimum-Norm method than to MUSIC. Proper spatial smoothing enables the performance of the Minimum-Norm method to be made comparable to MUSIC. Also, in the case of MUSIC though spatial smoothing improves the spectral efficiency factor, minimal smoothing is desirable for Root-Music. Computer simulations are provided to substantiate the analysis.

EDICS: 5.2, 5.2.2, 5.2.3

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STATE SPACE MODEL-BASED PARAMETER ESTIMATION* METHODS AND SOME APPLICATONS

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ABSTRACT

This paper reviews state space model-based methods for signal processing applications. A state space framework is shown to provide a convenient tool for exposing and exploiting structure inherent in many model based methods. It is also shown that there exist state space methods which are robust to noise in data, and to numerical errors. From a computational point of view, the methods are often less complex than existing competing methods. Futhermore they only involve matrix operations which are suitable for systolic/wavefront implementation.

I. INTRODUCTION

In this paper, state space model based methods for model based signal processing are reviewed [1-6]. Rational modeling of signals followed by estimation of the parameters of the model has been extensively discussed in the signal processing literature. Most often in this context, a direct form or transfer function parameterization of the models has been used. In this paper, state space representation of these models along with methods based on these respresentations are discussed. The key features of a state space based approach are summarized below.

1. The state space framework provides a convenient mechanism for exposing the structure that is present in model based methods.
2. Robust methods to estimate the state space parameters of these models are available. They result in model based signal processing methods that are numerically robust, and highly accurate even when noisy finite data records are used.
3. The computational complexity of these methods is usually less or comparable to existing competitive methods. Moreover, the operations involved are matrix oriented making them suitable for systolic/wavefront implementation.

II. STATE SPACE MODELING

In signal processing, digital filters or linear time invariant (LTI) systems with rational transfer functions have played a central role. Most often they are described using constant coefficient difference equations, i.e.

$$y(n) = \sum_{i=1}^{p} a_i\, y(n-i) + \sum_{i=0}^{p} b_i\, u(n-i) \tag{1}$$

where $u(n)$ is the input and $y(n)$ the output. In the transform domain the system is described by its transfer function $H(z)$ where

$$H(z) = \frac{A(z)}{B(z)}, \tag{2a}$$

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